INTERPRETATION OF REVERSE ALGORITHMS IN SEVERAL MESOPOTAMIAN TEXTS

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Is it possible to discuss proofs in texts which contain only numbers and no verbal element? I propose to analyze a Mesopotamian tablet containing a long series of reciprocal calculations, written as numeric data in sexagesimal place value notation. The provenance of this tablet, which today is conserved at the University Museum in Philadelphia under the number CBS 1215, is not documented, but there are numerous parallels from the scribal schools of southern Mesopotamia, notably Nippur and Ur, all from the Old Babylonian period (beginning two millennia before the Christian era). Thus, one might suppose that it shares in the scribal tradition inherited from the southern Sumero-Akkadian culture.[1] The text is composed of only two graphemes: vertical wedges (ones) and *Winkelhaken* (tens).[2] The limited number of graphemes is clearly not due to the limited knowledge of writing possessed by the author of the text. The tablet was composed at the time when 'the scribal art' (nam-dub-sar, in Sumerian) achieved its most refined developments, not only in the domains of mathematics and Sumerian or Akkadian literature, but also in the consideration of writing, language and grammar.[3] Hence, my hypothesis is that this text contains an original mathematical contemplation and that a close analysis of the tablet and its context yields the keys to understanding the text.[4]

Purely numeric texts are not rare among cuneiform documentation, but, with the exception of the famous tablet Plimpton 322 which has inspired an abundant literature, such texts have drawn relatively little attention from historians.[5] Indeed, the numeric tablets do not contain information written in verbal style (in Sumerian or in Akkadian language) and then numeric tablets are less explicit than other types of tablets in the intentions and the methods of their authors. It is generally admitted that numeric tablets are some sort of collection of exercises destined for pedagogical purposes. However, the content and context of the tablets show that the purposes of a text such as that of tablet CBS 1215 were greater than simple pedagogy. In particular, I would like to show in this article that the text is organized in order to stress the operation of the reciprocal algorithm and to show why the series of steps on which it is founded leads effectively to the desired reciprocal.

Before I go too far into the analysis, let me give a brief description of the tablet. The text is composed of 21 sections. (See the transcription in Appendix 1.) The entries of the sections are successively 2.5, 4.10, 8.20,…, 10.6.48.53.20, namely the first 21 terms of a geometric

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1 According to A. Sachs who published it, the tablet CBS 1215 is part of a collection called 'Khabaza 2', purchased at Baghdad in 1889. He thought it hardly possible that it came from Nippur, making reference to the intervening disputes among the team of archaeologists at Nippur (Sachs 1947: 230 and n. 14).

2 See the copy by Robson 2000: 23, and an extract of this copy in Table 12.3 below.

3 Cavigneaux 1989.

4 I thank all those who, in the course of seminars or through critical readings, have participated in the collective work of which this article is the result, beginning with Karine Chemla, whose remarks have truly improved the present version of the text.

5 On the subject of Plimpton 322, a tablet probably from the Old Babylonian period perhaps from Larsa, which presents a list of 15 Pythagorean triplets in the form of a table, see notably Robson 2001a, Friberg 2007: Appendix 7. Among the other analyses of numeric texts, outside that which bears upon the tablet studied here (Sachs 1947), one may cite those which concern the tables from the first millennium, such as the large table of reciprocals from the Seleucid period AO 6456–for example Bruins 1969 and Friberg 1983, and several other tables from the same period (Britton 1991-93, Friberg 2007, Appendix 8).

progression for an initial number 2.5 with a common ratio of 2. (Details on the cuneiform notation of numbers and their transcription appear later.) Other than the absence of any verbal element in its writing, the text possesses some obviously remarkable properties. (See Table 3 and Appendix 1.)

1) In each section, the numbers are set out in two or three columns. Thus, the spatial arrangement of the numeric data is an important element of the text.

2) The sections are increasingly long and, as will be seen, the result appears to be the application of iterations.

3) In each section, the last number is identical to the first. The procedure progresses in such a fashion that its point of arrival corresponds exactly with its point of departure. The text, therefore, reveals the phenomena of reciprocity.

What do these three properties (spatial arrangement, iteration and reciprocity) reveal to us? Do they disclose the thoughts of the ancient scribes about the mathematical methods which constitute the reciprocal algorithm, particularly about the topic of its validity? In order to respond to these questions, it will be necessary not only to analyze the text in detail, but also to compare and contrast it with other texts.

Reciprocal algorithms are not known only by their numeric form. In particular, a related tablet, VAT 6505, contains a list of instructions composed in Akkadian. Sachs[6] has shown that these instructions refer to calculations found in the numeric tablet CBS 1215. Thus, we have both a numeric text and a verbal text related to the same algorithm. These two texts both refer to the reciprocal algorithm in widely different manners. Neither do they employ the same means of expression, nor do they deliver exactly the same type of information. Thus there is a shift between the different texts and the practices of calculation to which they refer.

In addition, some properties of the tablet CBS 1215, notably those which concern spatial arrangement and reciprocity, are likewise manifested in calculations of square roots. Such is notably the case for the tablet UET 6/2 222, which is an Old Babylonian school exercise from Ur. (See Table 12.1 below) Also, in the case of the square root algorithm just as for the reciprocal algorithm, both numeric and verbal texts are attested. In fact, J. Friberg has shown that tablet IM 54472, composed in Akkadian, contains instructions which relate to calculations found in the numeric tablet UET 6/2 222.[7]

In order to facilitate the reading of the following sections, which alternate between different tablets, I have designated the tablets by the letters A through D. The concordance between these letters, their inventory numbers and provenance is presented in Table 12.1.

[6] Sachs 1947.
[7] Friberg 2000: 108-112.

	Inventory #	Provenance	Contents	Style
A	CBS 1215	Unknown	Reciprocal	Numeric
B	VAT 6505	Unknown	Reciprocal	Verbal
C	UET 6/2 222	Ur	Square Root	Numeric
D	IM 54472	Unknown	Square Root	Verbal

Table 12.1: Principal Texts Studied Here[8]

In addition, many other parallels to tablet A exist. In some cases, entire sections of the text are identically reproduced. Such reproductions and citations occur principally in the texts from the scribal schools which operated in Nippur, Ur and elsewhere.[9] The school texts yields precious information about the context of the use of the reciprocal algorithm and will be used on a case-by-case basis to supplement the small, essential body of texts presented in Table 12.1.[10]

The historical problem posed by relationships which may have existed between the authors of different texts is difficult to resolve because the provenance is usually unknown and the dating is uncertain. Some available information seem to indicate that the numeric texts and their pedagogical parallels may pertain to the southern tradition (Ur, Uruk, Nippur), and the verbal texts, notably Tablet B which may come from Sippar, belong to the northern tradition of Old Babylonian Mesopotamia.[11] The possible historic opposition between the north and the south, however, did not exclude certain forms of communication, since the two traditions were not isolated and the scribes from different regions had contact with one another through numerous exchanges, notably circulating school masters.[12] Even though uncertainty about the sources does not permit the establishment of a clear geographic distribution, it is entirely possible that the two types of texts existed in the same contexts. Regardless of the relationship between the authors of these different styles of texts, it is possible to hypothesize two points of view about the same algorithm. The important point, whether or not these two points of view emanate from the same scribal context, is that they clearly have different objectives. The verbal texts are series of instructions, which appear to have been intended to help someone execute the algorithm. Some portion of the numeric tables are school exercises intended for the training of student scribes. The function of Tablet A seems to have been of another nature.

Tablet A does not conform to the typology of a school tablet, even though it was used in an educational context, as was probably the case with all the mathematical texts of the Old Babylonian period. Through a comparison of tablet A with parallel or similar texts, I would like to provide more detailed responses to the questions concerning its function–Is Tablet A

[8] The tablets of Table 12.1 have been published in the following articles and works. A=CBS 1215 in Sachs 1947 for the transliteration and interpretation, Robson 2000: 14, 23-24 for the hand copy and several joins ; B=VAT 6505 in Neugebauer 1935-7 I: 270; II: pl. 14, 43; C=UET 6/2 222 in Gadd and Kramer 1966: 248; D=IM 54472: in Bruins 1954. Other than the tablet from Ur, the tablets come from illicit excavations. VAT 6505 may come from the north because of its orthographic and grammatical properties (H2002: 331 n. 383); according to Friberg 2000: 106, 159-160, it may come from Sippar. IM 54472 likewise may come from the north, perhaps from Shaduppum (Friberg 2000: 110).

[9] What are called 'school tablets' in Assyriology are the products of students in scribal schools. These tablets generally have a standardized appearance and contents, and because of this fact are easily recognizable, at least in the case of those which date from the Old Babylonian period.

[10] This documentation may be specified further: the list of parallels with A is presented in Table 12.6; the other tablets containing reciprocals are assembled in Table 12.7; those which contain calculations of square and cubic roots are in Table 12.8.

[11] The provenances of different tablets and their parallels are detailed in the notes relative to Tables 12.1, 12.6, 12.7, and 12.8. In the case of Mari, it is interesting to note that the tablets from this northerly site seem more akin to the tablets of the south than those of the north. Thus, if different scribal traditions were confirmed, they would clearly reveal complex trans-regional phenomena of communication, and not only local peculiarities.

[12] Charpin 1992, Charpin and Joannès 1992.

only a collection of exercises, from which the school exercises were extracted? What is the tablet's relationship to pedagogical practice? How does the information differ from the information presented in the verbal texts? What specific significance may be determined from its structure or its layout? These questions, as will be seen, are connected in the way that Tablet A corresponds with the operation of the reciprocal algorithm and with its justification.

PLACE VALUE NOTATION AND RECIPROCALS

Since numeric texts are constructed of numbers written in the sexagesimal place value notation characteristic of Mesopotamian mathematical texts, let us review the key principals of this notation. With the base being sixty, there are 59 ‘digits.’ (Zero is not found in the Old Babylonian period.) These 59 digits are represented by the repetition of the signs 1 (a vertical wedge) and 10 (the *Winkelhaken*) as many times as necessary.[13]

Examples: 𒈫 (2) 𒌋𒐈(13) 𒎙 (20)

According to the positional principle, each unit in a given place represents sixty units of the preceding place (at its right). For the transcription of numbers, I have followed the modern notation proposed by F. Thureau-Dangin, wherein the sexagesimal digits are separated by dots.[14]

Example: 𒈫𒌋𒐈𒎙 is rendered as 2.13.20

In cuneiform texts, no place is marked as being that of the units, thus the numbers have no value; they are determined to a factor 60^n (where n is some whole positive or negative number), which, after a fashion, resembles ‘floating decimal’ notation. For example, the numbers 1, 60, 60^2, 1/60 are all written in the same way, with a vertical wedge (𒁹): the scribes did not make use of any special signs such as commas or zeros in the final places similar to those we use in modern Indo-Arabic numerals. In the texts studied here, the operations performed on the numbers are multiplications, the determination of reciprocals and square roots, namely operations which do not require that the magnitudes of the numbers be fixed. In the transcriptions, translations and interpretations presented here, I have therefore not restored the orders of magnitude, in keeping with the indeterminacy of the value in the cuneiform writing. However, in these circumstances, might it be possible to establish ‘equalities’ between numbers, although their values are not specified? Even though the sign ‘=’ might be considered an abuse of language (and an anachronism), I use it in the commentary. This convenience seems acceptable to me insofar as we bear in mind that the sign ‘=’ denotes not a relationship of equality between quantities, but rather an equivalence between notations. For example, 2×30=1 signifies that the product of 2 and 30 is noted as 1.

How were these sexagesimal numbers used in calculations? The great number of school tablets discovered in the refuse heaps of the scribal schools present relatively accurate information about both the way in which place value notation was introduced in education in the Old Babylonian period and also its use. The course of the scribes' mathematical education is particularly well documented at Nippur, the principal centre of teaching in Mesopotamia.[15] At Nippur, and undoubtedly in the other schools, the first stage of mathematical apprenticeship consisted of memorizing many lists and tables: metrological lists

[13] The word ‘digit’ here indicates each sexagesimal place. These ‘digits’ are written in additive decimal notation.

[14] Other authors prefer to separate the sexagesimal places by a blank space or a comma (such is the case of Sachs, as will be seen later).

[15] Robson 2001, Robson 2002 and Proust 2007.

(enumerations of measures of capacities, weights, areas and lengths), metrological tables (tables of correspondence between different measures and numbers in place value notation) and numerical tables (reciprocals, multiplications and squares).[16] After having memorized these lists, the apprentice scribes used these tables in calculation exercises which chiefly concerned multiplication, the determination of reciprocals and the calculation of areas. Documentation shows that the place value notation came at precise moments in the educational curriculum. The place value notation does not occur among the expression of measurements which appeal to other numerations, based on additive principle. They appear in the metrological tables, where each measure (a value written in additive numeration followed by a unit of measure) is placed in relation to an abstract number (a number in place value notation, not followed by a unit of measure). Moreover, the abstract numbers[17] are found exclusively in the numeric tables and in exercises for multiplication and advanced calculations of reciprocals. The calculation of areas necessitates the transformation of measures into abstract numbers and back again, transformations assured by the metrological tables.[18]

Let us return to the topic of the determination of reciprocals, which is the subject of Tablet A. A small list of reciprocal pairs was memorized by the apprentice scribes in the course of their elementary education. These pairs form a standard table, found in numerous sources at Nippur and also in the majority of Mesopotamian educational centres. That table is as follows:

N	inv(*N*)	*N*	inv(*N*)	*N*	inv(*N*)
2	30	15	4	36	1.40
3	20	16	3.45	40	1.30
4	15	18	3.20	45	1.20
5	12	20	3	48	1.15
6	10	24	2.30	50	1.12
8	7.30	25	2.24	54	1.6.40
9	6.40	27	2.13.20	1.4	56.15
10	6	30	2	1.21	44.26.40
12	5	32	1.52.30		

Table 12.2: Standard Reciprocal Table

Obviously, the entries of the standard reciprocal table are the reciprocals of regular sexagesimal single place numbers, plus two reciprocals for numbers in two places (1.4 and 1.21).[19]

The determination of a reciprocal is an important operation for the scribes because the operation which corresponds with our division was effected through the multiplication by the reciprocal. Two consequences result from this conceptualization of 'division.' First, it privileges the regular numbers, which, in fact, are omnipresent in the school texts. Next,

[16] These tables are described in detail in Neugebauer 1935-7 I: ch. I.

[17] In the following pages, 'abstract numbers' will refer to the numbers written in sexagesimal place value notation.

[18] For more details about these mechanisms, see Proust 2008.

[19] Two numbers form a reciprocal pair if their product is written as 1. A regular number in base-60 is a number for which the reciprocal permits a finite sexagesimal expression (numbers which may be decomposed into the product of factors 2, 3 or 5, the prime divisors of the base). The oldest reciprocal tables contain not only the regular numbers, but also the complete series of numbers in single place (1 to 59). In these tables, the irregular numbers are followed by a negation: 'igi 7 nu', meaning '7 has no reciprocal; see for example the two Neo-Sumerian reciprocal tables known from Nippur, HS 201 in Oelsner 2001 and Ni 374 in Proust 2007: § 5.2.2. It may be said that although the Sumerian language contains no specific term to indicate the regular numbers, it nonetheless contains an expression for the irregular numbers: 'igi … nu'.

division is not properly identified as an operation. In order to effect a division, first a reciprocal is found, then a multiplication is made.[20] In this way, division has no name in Sumerian, contrary to the determination of a reciprocal (igi) and multiplication (a-ra$_2$).

The determination of the reciprocal of a regular number is thus a fundamental objective of Babylonian positional calculation. The standard tables furnish the reciprocals of the ordinary regular numbers. In what follows, I call the numbers which appear in Table 12.2 'elementary regular factors.' For the other regular numbers which do not appear in the standard table, the scribes had recourse to a reciprocal algorithm, which is precisely what Tablet A handles.

Sachs identified the reciprocal algorithm thanks to the verbal text of Tablet B (VAT 6505).[21] First, I present the way in which Sachs understood this algorithm and described it in an algebraic formula. Then, I will analyze the way in which Tablets A and B both refer to the same algorithm and the ways in which they differ. This contrast will indirectly permit some of the particular objectives pursued in Tablet A to be clarified.

SACHS' FORMULA

The colophon of Tablet B indicates that the text is composed of twelve sections. The entries are the first twelve terms of a geometric series for an initial number 2.5 with a common ratio of 2–the same terms which constitute the beginning of Tablet A. In fact, only five sections are even partially preserved but these remains allowed Sachs to reconstitute the entirety of the original text. The well-preserved entry of the seventh section is 2.13.20, that is 2.5 after six doublings. The text may be translated as follows:

B #7 translation by Sachs 1947: 226[22]	
1.	2,[13],20 is the *igûm*. [What is the *igibûm*?]
2.	[As for you, when you] perform (the operations),
3.	take the reciprocal of 3,20 ; [you will find 18]
4.	Multiply 18 by 2,10; [you will find 39]
5.	Add 1; you will find 40.
6.	Take the reciprocal of 40; [you will find] 1,30.
7.	Multiply 1,30 by 18,
8.	you will find 27. The *igibûm* is 27.
9.	Such is the procedure.

According to Sachs, whose notations I have reproduced,[23] the algorithm is based on the decomposition of the initial number c as the sum $a+b$, this decomposition is summarized by the following formula (in which the reciprocal of a number n is denoted by $\bar{n}$) :

[20] The concept of division presented here is that which was taught in the scribal schools and the one used most often in mathematic texts, particularly in those texts found in the present article. However, this is not the only extant conceptualization. For example, divisions by irregular numbers occur frequently, but they are formulated as problems: find the number, which, when multiplied by some number, returns some other number (H2002: 29). Likewise, among the mathematical texts, there exist slightly different usages of 'reciprocals', somewhat closer to our concept of fractions. In certain texts, the goal is to take the fraction 1/7 or 1/11 of a number. (See, for example, the series of problems such as A 24194.) Finally, in rare cases, approximations for the reciprocals of irregular numbers are found (H2002: 29, n. 50).

[21] Sachs 1947.

[22] Damaged portions of text are placed in square brackets. *igûm* and *igibûm* are Akkadian words for pairs of reciprocals.

[23] In **translations,** like Neugebauer, Sachs used commas to separate sexagesimal digits, but unlike Neugebauer, he did not use 'zeros' and semicolons to indicate the order of magnitude of the numbers. He used these marks only in the mathematical commentaries and interpretations of the sources.

$$\bar{c} = \overline{a+b} = \bar{a}.\overline{(1+b\bar{a})}$$

Applied to the data in B #7, this formula leads to the following reconstruction:[24]

$$c = 2,13;20$$

$$c = a + b = 3;20 + 2,10$$

$$\bar{a} = \overline{3;20} = 0;18$$

$$\bar{a}b = 0;18 \times 2,10 = 39$$

$$1 + \bar{a}b = 1 + 39 = 40$$

$$\overline{1+\bar{a}b} = \overline{40} = 0;1,30$$

$$\bar{c} = \bar{a} \times \overline{1+\bar{a}b} = 0;18 \times 0;1,30 = 0;0,27$$

On the one hand, the ‘Sachs' formula’ allows us to follow the sequence of calculations by the scribe and on the other hand it establishes for us the validity of the algorithm according to modern algebra. Moreover, it provides historians a key to understanding Tablet A and its numerous parallels. In fact, as indicated above, the first twelve sections of Tablet A contain the same numeric data as their analogs in Tablet B. For example, the transcription of Section 7 of Tablet A is as follows:

A #7	
2.13.20	18
40	1.30
27	2.13.20

In Tablet A #7 are found, in the same order, the numbers which appear in the corresponding section of Tablet B. Clearly, the numeric Tablet A refers to the same algorithm as the verbal text of Tablet B. Until now, ‘Sachs' formula’ has provided a suitable explanation of the reciprocal algorithm. This formula is generally reproduced by specialists in order to explain texts referring to this algorithm in numeric versions (Tablet A and its school parallels) as well in verbal version (Tablet B). (See Tables 12.6, 12.7 and 12.8.) However, in my estimation, this formula does not permit us to explain the differences between the texts A and B, nor grasp specific objectives pursued by them in referring to the algorithm. The principal shifts that I note between the ‘Sachs' formula’ and the texts which it supposedly describes are the following:

1) The tools employed by Sachs in his interpretation (algebraic notation, using semicolons and zeros) are not those used by the Old Babylonian scribes. The ‘Sachs' formula’ leaves unclear the actual practices of calculation to which the text of Tablets A and B makes reference.

2) The text of Tablet B, just like the remains of Tablet A, does not refer to the algorithm in an abstract manner but in a precise manner, with a series of particular numbers, namely 2.5 and its successive doublings. The algebraic formula does not explain the choice of these particular numbers.

[24] Ibid.: 227.

3) None of the properties of Tablet A (spatial arrangement, iteration, and reciprocity) are found in Tablet B. The 'Sachs' formula' does not allow the stylistic differences which separate Tablets A and B to be described or interpreted.

I would like to draw attention to the fact that Tablet A tells us much more than an algebraic formula in modern language can convey. What information is conveyed by the text of Tablet A but not contained by the 'Sachs' formula?' Answering this question will help us understand the original process of the ancient scribes and their methods of reasoning. In that attempt, I will concentrate for now on the particular properties of the text of Tablet A, then on the particular numbers found therein.

SPATIAL ARRANGEMENT

Using Sachs' interpretation as a starting point, I am ready to detail the algorithm of determining a reciprocal to which Tablet A refers. I rely on with the numeric data in A #7, which are presented above and in appendix 1:

- the number 2.13.20 terminates with 3.20, which appears in the reciprocal table, thus 3.20 is an elementary regular factor[25] of 2.13.20;

- the reciprocal of 3.20 is 18; 18 is set out on the right;

- the product of 2.13.20 by 18 is 40; 40 is therefore a second factor and it is regular; 40 is set out on the left and its reciprocal 1.30 is set out on the right;

- the number 2.13.20 is therefore factored into the product of two elementary regular factors: 3.20 and 40;

- the reciprocal of 2.13.20 is the product of the reciprocals of these two factors, namely the numbers set out on the right: 1.30 and 18;

- the product of 1.30 by 18 is 27

- 27 is the desired reciprocal.

Then, the reciprocal of this result is found, leading back to 2.13.20, the same number as the initial data. For the time being, let us put aside this last step in order to comment on the reciprocal algorithm, as I have reconstituted it in the steps above.

Essentially, the algorithm is based on two rules. On the one hand, a regular number can always be decomposed into the product of elementary regular factors–that is, into the product of numbers appearing in the standard reciprocal table.[26] On the other hand, the reciprocal of a product is the product of reciprocals. These rules correspond to the spatial arrangement of the numbers into two columns.

The factorization of 2.13.20 appears in the left column:

2.13.20 = 3.20 × 40

The factorization of the reciprocal appears in the right column:

18 × 1.30 = 27

Let us note an interesting difference between Tablets A and B in their manner of executing the procedure. No addition appears in Tablet A, but one instance appears in Tablet B (line 5). This addition may be interpreted as being a step in the multiplication of 2.13.20 by 18. The

[25] As indicated above, I call any factor which appears in the standard reciprocal table (that is, Table 2) a 'elementary regular factor'.

[26] Naturally, this decomposition is not unique. The choices made by the scribes will be analyzed later.

number 2.13.20 is decomposed into the summation of 2.10 and 3.20. Then each term is multiplied separately by 18, and finally the two partial products are added. This method of multiplication is economical. With one of the partial products being obvious (3.20×18 is equal to 1 by construction), the multiplication is reduced to 2.10×18. This decomposition of multiplication may draw on the practices of mental calculation or the use of an abacus. It therefore seems that the instructions of text B refer not only to the steps of the algorithm, but also to the execution of multiplications. Text A, on the contrary, makes reference only to the steps of the algorithm. The execution of multiplications seems to outside the domain of text A. I will return later to this external aspect of multiplication in relation to the analysis of errors.

Finally, let us underscore that the spatial arrangement of the text on Tablet A does not correspond to the normal rules of formatting tablets in the scribal tradition. When the scribes wrote on tablets, they were accustomed to starting the line as far left as possible and ending it as far right as possible, even if it meant introducing large spaces into the line itself. This method of managing the space on the tablet is found in all genres of texts – administrative, literary and mathematic. The example on the obverse of tablet Ni 10241 (see the copy in Appendix 2) is a good illustration of this. In this tablet, the last digit of the number contained in each line is displaced to the right and a large space separates the digits 26 and 40 in the number 4.26.40. The same happens with the digits 13 and 30 in the number 13.30. This space has no mathematical value. It corresponds to nothing save the rules of formatting. The management of spaces in Tablet A, and likewise the reverse of Tablet Ni 10241, is different. The spaces there have a mathematical meaning, since they allow columns of numbers to appear. The areas of writing to the left, centre and right have a function with respect to the algorithm.

Thus appear in Tablet A the principles of the spatial arrangement of numbers which have a precise meaning in relation to the execution of the reciprocal algorithm. In each section, certain numbers (the factors of the number for which the reciprocal is sought) are placed to the left; others (the factors of the reciprocal) are set out on the right; and still others (the products of the factors) are located in the central position. A simple description of these principles of spatial arrangement suffices to account for the basic rules on which it is based. Every regular number may be decomposed into products of elementary regular factors, and the reciprocal of a product is the product of the reciprocals. More than an algebraic formula, this explanation of the principles of spatial arrangement allows us to understand the working of the algorithm and to reveal some elements of what might have been the actual practices of calculation.

The calculations to which the different results appearing in the columns correspond are multiplications. There is, in this text, a close relationship between the floating place value notation and multiplication, just as in the body of school documentation. However, if the text records the results of multiplications, it bears no trace of the actual execution of these operations, whereas such traces seem detectable in the verbal text of Tablet B as said above. In Tablet A, in contrast, the steps of the algorithm and the execution of multiplication are disassociated.

Even though the texts of the Tablets A and B refer to the same algorithm, some features distinguish them. In the first case, the text is two-dimensional: the spatial arrangement of the numbers plays a critical role, referring to the steps of calculation but not to the manner of carrying out the multiplications. In the second case, the text concerns a linear continuation of the instructions, which refer not only the algorithm, but also to the execution of the multiplications. Another difference appears in section 5. When the numbers for which the reciprocal is determined reach a certain size, the phenomenon of iteration appears in Tablet A, but not in Tablet B (so far as the preserved portion allows us to judge).

ITERATION

Let us consider Section 20 of which the transcription and the copy are given in Table 12.3 below. (The bold type and underscoring have been added.) First, I will explain the first part of the section, concerning the reciprocal of 5.3.24.26.40 (lines 1 to 9).

A #20

Line #	Transcription		
1	5.3.24.2**6.40**		[9]
2	45.30.**40**		1.30
3	1.8.**16**		3.45
4	4.**16**		3.45
5	**16**		3.45
6		14.3.45	
7		5[2.44].3.45	
8		1.19.6.5.37.30	
9	<u>11.51.54.50.37.**30**</u>		2
10	23.43.49.41.**15**		4
11	1.34.55.18.**45***		16
12	25.18.**45***		16
13	**6.45**		1.20
14	**9**		6.[40]
15		8.53.20	
16		2.22.13. 20	
17		37.55.33.20	
18		2.31.42.13.20	
19		<u>5.3.24.26.40</u>	

Copy Robson 2000: 23

Figure 12.1

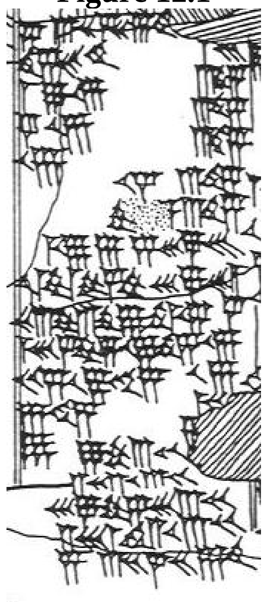


Table 12.3: Transcription and Copy of Section 20

The idea of determining the reciprocal through factorization is used with more force here. The number for which the reciprocal is sought is 5.3.24.26.40. The first factor chosen is 6.40, the last part of the number. Its reciprocal is 9 (written to the right). The product of 5.3.24.26.40 and 9 is 45.30.40 (written to the left). The reciprocal of this number is not given in the standard reciprocal tables, thus once again the same sub-routine is applied. The process continues until an elementary regular number is obtained. In the fourth iteration, 16 is finally obtained. With the reciprocals having been written down in the right-hand column at each step, it suffices to multiply these numbers to arrive at the desired reciprocal. The multiplication is carried out term by term,[27] in the order of the group of intermediate products in the central column. In other words, 3.45 is multiplied by 3.45. The result (14.3.45) is multiplied by 3.45. Then that result is multiplied by 1.30; and that result is multiplied by 9. Thus for 11.51.54.50.37.30 the desired reciprocal is obtained.

In modern terms, the algorithm may be explained by two products:

The factorization of 5.3.24.26.40 appears in the left-hand column (or, more precisely in the last part of the numbers in the left-hand column):

5.3.24.26.40 = 6.40 × 40 × 16 × 16 × 16

Likewise, the factorization of the reciprocal appears in the right-hand column:

9 × 1.30 × 3.45 × 3.45 × 3.45 = 11.51.54.50.37.30

[27] In the cuneiform mathematical texts, multiplication is an operation which has no more than two arguments.

Since the sub-routine is repeated, the usefulness of the rules for spatial arrangement of the text becomes clear. The factors of a number for which the reciprocal is sought are on the left. The factors of the reciprocal are on the right and the partial products are in the centre. The spatial arrangement of the text probably corresponds with a practice allowing an automatic execution of the sequence of operations. Such an arrangement displays the power of the algorithm and demonstrates possibilities of the spatial organization of the writing – possibilities that the linear arrangement of a verbal text like Tablet B does not include.

REVERSE ALGORITHMS

Now let us consider the entirety of Section 20 of Tablet A (Table 12.3 above). Lines 1 through 9 show step-by-step that the reciprocal of 5.3.24.26.40 is 11.51.54.50.37.30. This number, in turn, is set out on the left and subjected to the same algorithm: 11.51.54.50.37.30 ends with 30; the reciprocal of 30, which is 2, is set out on the right, etc. As in the other examples, the number 11.51.54.50.37.30 is decomposed into the product of elementary regular factors. The reciprocals of these factors are set out on the right, and finally the reciprocal is obtained by multiplying term-by-term the factors set out on the right. The result is, naturally, the initial number, 5.3.24.26.40. It is the same in all the sections: after having 'released'[28] the reciprocal in terms of a quite long calculation, the scribe undertakes the determination of the reciprocal of the reciprocal by the same method and returns to the point of origin. Each section is thus composed of two sequences: the first sequence, which I will call the direct sequence and the second sequence, the reverse of the first (in the sense that it returns to the point of departure). In what way did this scribe execute the algorithm in the reverse sequence? What interest did he have in systematically undoing what he had done?

To execute the reverse sequence, the scribe would have been able to use the results of the direct sequence, which provided him with decomposition into elementary regular factors. It was enough for him to consider the factors set out on the right in the first part of the algorithm. For example in Section 20, to find the reciprocal of 11.51.54.50.37.30, he was able to select the factors 3.45, 3.45, 3.45, 1.30 and 9 which appeared in the first part, but this simple repetition of factors was not what he did. He applied the algorithm in its entirety, and as in the direct sequence, the factors were provided by the final part of the number. (In 11.51.54.50.37.30, the first elementary factor is 30, then 15, etc.) This same algorithmic method is applied in the direct sequence and in the reverse sequence of each section. I will elaborate on this point later, particularly when analyzing the selection of factors in the collection of texts. Already this remark suggests a first response to the question of the function of the reverse sequence. It might be supposed that the reverse sequence is intended to verify the results of the direct sequence, but if such were the case, it would be expected that the scribe would chose the most expedient method, and the most economic in terms of calculations. Clearly, he did not search for a shortcut. He did not use the results provided from his previous calculations, which could have been done in several ways. As has just been seen, he could have used the factors already identified in the direct sequence. It would also have been simple for him to use the reciprocal pairs calculated in the preceding section. Section 19 establishes that the reciprocal of 2.31.42.13.20 is 23.43.49.41.15. However, several texts attest to the fact that the scribes knew perfectly well that when doubling a number, the reciprocal is divided by 2 (or, more exactly, its reciprocal is multiplied by 30).[29] In verifying

[28] The Sumerian verb which designates the act of calculating a reciprocal is du$_8$ (release) and the corresponding Akkadian verb is *paṭārum*. F. Thureau-Dangin translates this verb as 'dénouer,' and J Høyrup as 'to detach'.

[29] Some texts containing lists of reciprocal pairs founded on this principle are known: beginning with a number and its reciprocal, they give the following doublings and halvings. For example the tablet from Nippur N 3958 gives the series of doublings/halvings of 2.5 / 28.48 (Sachs 1947: 228).

the result of Section 20, it was therefore sufficient to multiply 23.43.49.41.15 by 30. Proceeding in another way, the scribe could have multiplied together the initial number and its reciprocal in order to verify the fact that the product was equal to 1. These simple methods show that it was unnecessary to reapply the reciprocal algorithm. In fact, the reverse sequence does not seem to have had the verification of the result of the direct sequence as a primary purpose. The fact that, in the second part, the algorithm was used in its entirety provokes speculation that if it were a verification, it concerns the algorithmic method itself and not merely the results which it produced.

Another important aspect of the algorithm is the selection of particular numbers. This aspect appears in comparison between Tablet A and B. Both use the same geometric series. The particular role of this series, omnipresent in all Mesopotamian school exercises of the Old Babylonian period, is one of the first points which ought to be made clearer. A second point is connected to the algorithm itself. Given that the decomposition into the product of elementary regular factors not being unique, one wonders if some rule governed the scribes' choice of one factor over another. This question invokes another question, even more interesting in light of the questions discussed in this article: did the scribes applied different rules to select factors in the direct and reverse sequences? Does this selection clarify the function of the reverse sequences?

NUMERIC REPERTORY

As has been seen, the entries in the sections of Tablet A, as with those of B, are the terms of the geometric progression for an initial number 2.5 with a common ratio of 2. What information did the scribe obtain in each of these sections? After the reciprocal of 2.5 had been obtained by factorization, it is possible to find all the other reciprocals by more direct means, as has been explained above. For example, in each section, the reverse sequence could repeat the calculations of the direct sequence, since it leads back to the point of departure, but this is not the case. The repeated application of the reciprocal algorithm does not produce any new result (other than the reciprocal of 2.5). From the perspective of an extension of the list of reciprocal pairs, this text is useless. Thus, what is the function of the repetition of the same algorithm forty-two times (in 21 sections, each one containing a direct sequence and a reverse sequence), since it returns results already seen?

First of all, why has the scribe chosen the number 2.5, the cube of 5, as the initial number of the text? This selection undoubtedly has some importance, because the entry 2.5 and the terms of the dyadic series which result provide the majority of numeric data in exercises found in the school archives of Mesopotamia. An initial explanation could be drawn from the arithmetic properties of this number. It has been seen previously that the list of entries in the standard reciprocal table (Table 12.2) is composed of regular numbers in a single place, followed by two more numbers in two places, 1.4 and 1.21. However, we note that 1.4, 1.21 and 2.5 are respectively powers of 2, of 3 and of 5 ($1.4=2^6$; $1.21=3^4$; $2.5=5^3$). Better yet, if the list of all the regular numbers in two places is set in the lexicographic order,[30] the first number is the first power of 2, that is, 1.4; the first power of 3, that is, 1.21, comes next, and the first power of 5, 2.5, comes thereafter. Thus, in some ways, 2.5 is the logical successor in the series 1.4, 1.21. Even if this explanation is thought too speculative, one must admit the privileged place accorded to the numbers 1.4, 1.21 and 2.5. The importance of the powers of 2, of 3, and of 5 perhaps indicates the manner by which the list of regular numbers (and their

[30] The numbers cannot be arranged according to magnitude, since this is not defined. The school documentation shows that in some cases the scribes used a lexicographical order. See for example the list of multiplication tables. Here, reference is made to this lexicographical order. The numbers are set out in increasing order by the leftmost digit, then following, etc.

reciprocals) were obtained. Beginning with the first reciprocal pairs, the other pairs can be generated by multiplications by 2, by 3 and by 5 (and their reciprocals by multiplication by 30, 20 and 12 respectively). This process theoretically would allow the entire list of regular numbers in base-60 and their reciprocals to be obtained.[31] The importance of the series of doublings of 2.5 in the school documentation could also be explained by its pedagogical advantages. I will return to this point later.

For now, let us try to draw some conclusions by analyzing the selection of factors in the factorization procedure. The execution of the factorization depends, in each step, on the determination of the factors for the number for which the reciprocal is sought. Does the selection of these factors correspond to fixed rules? First of all, let us note that in all of Tablet A, the same choices of the factors correspond to identical numbers. For example, the number 1.34.55.18.45 appears several times, and in each case, the factor chosen is 3.45. Let us now examine these selections, by distinguishing between the case of the direct sequences (Table 12.4) and the reverse sequences (Table 12.5). The factorizations which present irregularities (in a meaning to be specified later) are presented in gray and numbered in the right of the tables. The factorizations are ordered according to Column 3, which contains the factors chosen in the different decompositions. Column 5 gives the largest elementary regular factor if it is different from the factor chosen by the scribe. Column 2 specifies the section to which the appropriate decomposition belongs (I considered only sections preserved enough to permit a safe reconstitution of the text).

[31] I think that reciprocal tables such as the one found in the large Seleucid tablet AO 6456 were constructed in this way. A similar idea is developed by Bruins 1969.

Direct Sequences

Number to Factor	Section	Factor Chosen	Reciprocal of Factor	Largest Elementary Regular Factor	
2.**5**	#1	5	12		
4.10	#2	10	6		
4.16	#18, 20, 21	16	3.45		
1.8.**16**	#20, 21	16	3.45		
8.20	#3	20	3		
10.**40**	#11, 12	40	1.30		
2.50.**40**	#15	40	1.30		
45.30.**40**	#20	40	1.30		
4**2.40**	#13, 14	2.40	22.30	40	
11.2**2.40**	#18	2.40	22.30	40	(1)
3.2.**2.40**	#21	2.40	22.30	40	
3**3.20**	#5	3.20	18		
2.1**3.20**	#7	3.20	18		
8.5**3.20**	#9	3.20	18		
35.3**3.20**	#11	3.20	18		
2.22.1**3.20**	#13	3.20	18		
9.28.5**3.20**	#15	3.20	18		
10.6.48.5**3.20**	#21	3.20	18		
1**6.40**	#4	6.40	9		
1.**6.40**	#6	6.40	9		
4.2**6.40**	#8	6.40	9		
17.4**6.40**	#10	6.40	9		
1.11.**6.40**	#12	6.40	9		
4.44.2**6.40**	#14	6.40	9		
18.57.4**6.40**	#16	6.40	9		
1.15.51.**6.40**	#18	6.40	9		
5.3.24.2**6.40**	#20	6.40	9		

Table 12.4: Selection of Factors in the Direct Sequences

Reciprocal Sequences

Number to Factor	**Section**	**Factor Chosen**	**Reciprocal of Factor**	Largest Elementary Regular Factor	
7.**12**	#3	12	5		
1.41.**15**	#11	15	4	1.15	(2)
23.43.49.41.**15**	#19	15	4	1.15	
2.**15**	#5	15	4		(2’)
14.**24**	#2	24	2.30		
3.22.**30**	#10	30	2	2.30	
12.39.22.**30**	#14	30	2	2.30	
47.27.39.22.**30**	#18	30	2	2.30	(3)
50.37.**30**	#12	30	2	7.30	
11.51.54.50.37.**30**	#20	30	2	7.30	
13.**30**	#8	30	2		(3’)
3.**36**	#4	36	1.40		
6.**45**	#9	45	1.20		
1.**48**	#5	48	1.15		
28.**48**	#1	48	1.15		
25.18.**45**	#13	3.45	16	45	
1.34.55.18.**45**	#17	3.45	16	45	(4)
5.55.57.25.18.**45**	#21	3.45	16	45	

Table 12.5: Selection of Factors in the Reciprocal Sequences

Tables 12.4 and 12.5 show that the chosen factor is determined by the last digits of the number to be factored. In so doing, the scribes made use of an arithmetical property of the base-60 place value notation –that is, the numbers to be factorized are all regular and thus they always end with a sequence of digits which form a regular number.[32] All that is needed is to adjust for a suitable sequence. (In the case of 2.13.20, we may take 20, or 3.20, or even 13.20). In practice, the final part, insofar as it is an elementary regular number, is likely to be a factor. (For 2.13.20, the factor might be 20, or 3.20.) In the majority of cases, the scribe chose, from among the possible factors, the 'largest' (3.20 rather than 20), in order to render the algorithm faster.[33] Thus, in general, the selected factor is the largest elementary regular number formed by the terminal part of the number.[34] Nevertheless, this rule allows four exceptions (cases numbered in the last column of Tables 12.4 and 12.5), that need to be considered.

(1) – The selected factor, 2.40, does not appear in the standard reciprocal tables, and it is the factor 40 which ought to have been chosen. Nonetheless let us note that the reciprocal of 2.40 is 22.30, which is a common number which figures among the principal numbers of the standard multiplication tables. (The table of 22.30 is one of those learned by heart in the primary level of education, especially at Nippur.) Thus, 2.40 is 'nearly' elementary, and its reciprocal was undoubtedly committed to memory –so case (1) does not truly constitute an irregularity.

(2) and (3) – In case (2), the largest elementary factor is 1.15, but the factor 15, the entry of (2'), is used instead. In case (3), the selected factor could be either 2.30 or 7.30, but the factor 30, the entry of (3'), is used instead. This choice occurred as if the scribe sought to restrict the factors used in the calculation. The general rule of the 'largest elementary regular factor', regularly applied in the direct sequences, is, in the reciprocal sequences, opposed by another rule restricting the numeric repertory.

(4) – In this case, the factor might have been 45, but the scribe has obviously tried to use a larger factor. However, the numbers derived from the last two sexagesimal places (18.45 or 8.45) are not regular. Thus, 8 is decomposed into the summation 5+3, and the final part of the number selected as a factor is 3.45.

Several general conclusions may be drawn from these observations. First, the number of factors occurring in the decompositions is limited. They are principally 3.20 and 6.40 (less frequently 10, 16, 25, 40 and 22.30) for the direct sequences and principally 30 (less frequently 12, 15, 24, 36 and 45, 48 and 3.45) for the reverse sequences. This limited number of factors is explained by the way in which the list of entries was constructed–namely, 2.5, a power of 5, is multiplied by 2 repeatedly, giving rise to a series of numbers for which the final sequences describe regular cycles. However, the scribes' choices intervene. On the one hand, the direct sequences obey the 'greatest elementary regular factor' rule. On the other hand, the reverse sequences present numerous irregularities in regard to this rule. The number of factors used in the calculations is reduced. Finally, an interesting point to emphasize is that although the direct and reverse sequences refer to the same algorithm, they do not seem to share in the

[32] This property is the result of a more general rule: for a given base, the divisibility of an integer by the divisors of the base is seen in the last digits of the number. For a discussion of the particular problems resulting from divisibility in 'floating' base-60 cuneiform notation, a system in which there is no difference between whole numbers and sexagesimal fractions, see Proust 2007: § 6.2.

[33] The word *large* has nothing to do with the **magnitude** of the abstract numbers, since magnitude is not defined, but with their **size**. A two-places number is 'larger' than a single place number; for numbers with the same number of digits, the 'larger' number is the last in the lexicographical order. The speed of the algorithm depends on the size of the numbers thus defined: the 'larger' the factors are, the fewer factors there will be and thus fewer iterations. Let us specify that the order according to the size of the numbers is different from the lexicographical order mentioned above. The two orders appear in cuneiform sources. The order according to the size appears in the Old Babylonian reciprocal tables, and the lexicographical order occurs in the Seleucid reciprocal tables such as AO 6456, as well as in the arrangement of the multiplication tables in the Old Babylonian numerical tables.

[34] For this reason, Friberg 2000: 103-105 designates this procedure the *trailing part algorithm*.

same way the liberty permitted by the fact that the decomposition of numbers into regular factors is not unique. How do these two different ways of choosing the decomposition clarify the function of the reverse algorithm for us? Part of the answer is found in the school documentation. I will return to this question after analyzing the parallels with Tablet A.

The observation of errors appearing in this tablet brings something else to light. The fact that these errors are not numerous shows the high degree of erudition of the author of the text. Appearing in the transcription of A. Sachs and the copy of E. Robson, these errors are as follows:

- #4: the scribe has written 15.40 in place of 16.40.
- #5: the scribe has written 9 in place of 8.
- #11: the scribe has written 3**5.3**3.20 in place of 3**6.2**3.20.
- #19: the scribe has written 19 in place of 18.

The errors are all of the same type: forgotten or superfluous signs. The absence of a vertical wedge in certain instances, for example in section 4, may be the result of the deterioration of the surface of the tablet, not an error. In fact, in clay documents, signs are frequently hidden by particles of dirt or salt crystals, or flakes of clay have been broken off due to both ancient and modern handling.[35] Whatever the case may be, if the errors exist, they are not the result of errors in calculation, but simple faults in writing. Moreover, and this detail has great significance, the errors are not propagated in the following sequence of calculations.[36] The arithmetic operations themselves, namely the multiplications, are then carried out in another medium in which the error had not occurred. The text proceeds as if it does nothing but receive and organize the results of calculations computed in this external medium. For example, the fact that, in the number 36.23.20 of Section 11, the scribe has transformed one ten in the middle place into a unit in the left-hand place may be explained as an error in transferring a result from some sort of abacus. Quite probably, some of the multiplications, particularly those which appear in the last sections and involve big numbers, required outside assistance, probably in the form of a physical instrument (such as an abacus).

COMPUTING RECIPROCALS IN SCHOOL TEXTS

Tablet A possesses numerous parallels, nearly all of which appear in the characteristic form of tablets called Type IV by Assyriologists. Scribes used these Type IV tablets to train in numeric calculation. The copy presented in Appendix 2 is typical of these small lenticular or square tablets. Consideration of these parallels allows us to establish our tablet in the context of the scribal schools. This *corpus* in particular will allow us to determine the elements which relate directly to the school education to be detected, as well as those which do not seem to be connected to purely pedagogical purposes. From these comparisons, hypotheses about the function of the tablet, the reciprocal algorithm, and most notably the direct and reverse sequences may be put forth.

Let us consider all the known Old Babylonian tablets containing non-elementary reciprocal pairs (other than those which figure in the standard tables). To my knowledge, this set comprises a small group of about thirty tablets, listed in two tables (Tables 12.6 and 12.7)

[35] See the description of the state of this tablet by Sachs 1947: 230.

[36] It is not always the case in this genre of text. For example, in the tablet MLC 651, a school tablet in which the reciprocal is determined 1.20.54.31.6.40 (a term from the series of doublings of 2.5, see Table 12.4), an error appears in the beginning of the algorithm and propagates throughout the following text. The error is a real error in calculation, which arose in the course of the execution of one of the multiplications.

below.[37] In the first table, I have gathered the parallels of Tablet A. In the second table are found the other texts; they also contain reciprocal pairs extract from geometric series. The different columns of the tables provide information about the following points:

1- The inventory number and type of school tablet.

2- The provenance.

3- Reciprocal pairs contained in the tablet; when there are several pairs, the entries are always the terms of a geometric series with a common ratio of 2; I have indicated only the number of pairs and the first pair.

4- The format of the text, indicated by numbers: (1) if the text appears as a simple list of reciprocal pairs; (2) if the presence of a factorization algorithm is noted; (3) if the presence of direct and reverse sequences of the factorization algorithm is noted.[38]

5- In Table 12.6, a supplementary column indicates the corresponding section of Tablet A. Sections which have more than 20 doublings of 2.5 are called 'extrapolations.' Since Tablet A is limited to 20 doublings, these sections do not appear there.

Number, type	**Provenance**	**Contents**		**A**
		Reciprocal pairs	**Format**	
2N-T 496, IV	Nippur	16.40 / 3.36	(1)	A #4
3N-T 605, IV	Nippur	4.26.40 / 13.30	(1)	A #8
2N-T 115, IV	Nippur	9.28.53.20 / 6.19.41.15	(1)	A #15
Ni 10244, IV	Nippur	1.15.51.6.40 / 47.27.39.22.30	(1)	A #18
Ni 10241, IV	Nippur	4.26.40 / 13.30	(1) and (2)	A #8
2N-T 500, IV	Nippur	17.46.40 / 3.22.30	(1) and (2)	A #10
CBS 10201	Nippur	8 pairs : 2.5 / 28.48, etc.	(2)	A #1-8
N 3891, IV	Nippur	8.53.20 / 6.45	(1) and (3)	A #9
3N-T 362, IV	Nippur	17.46.40 / 3.22.30	(3)	A #10
UET 6/2 295, IV	Ur	2.5 / 28.48	(2)	A #1
W 16743ay, IV	Uruk	10.6.48.53.20 / 5.55.57.25.18.45	(1)	A #21
TH99-T196, IV	Mari	5.55.57.25.18.45 / 10.6.48.53.20	(1)	A #20 (reverse sequence)
TH99-T192, IV	Mari	1.9.26.40 / 51.50.24 Note: 1.9.26.40 = 8.20×8.20 and 8.20 is the entry of A #3.	(1)	A #3 (indirectly)
FLP 1283, IV	Unknown	Obverse: proverb; reverse: 2.5 / 28.48	(1)	A #1
YBC 10802, IV	Unknown	2.22.13.20 / 25.18.45	(1)	A #13
BM 80150	Unknown	Numeric table; 13 pairs: 2.5 / 28.48, etc.	(1)	A #1-13
MLC 651, IV	Unknown	1.20.54.31.6.40 / 44.29.40.39.50.37.30 (with an error in calculation)	(2)	A #23 (extrapolation)
YBC 1839, IV	Unknown	4.26.40 / 13.30	(2)	A #8
MS 2799, IV	Unknown	2.41.49.2.13.20 / 22.14.50.19.55.18.45	(1)	A #24 (extrapolation)

Table 12.6: Parallels with Tablet A

[37] The tablets cited in the Tables 12.4 and 12.5 have been published in the following articles and works. CBS 10201 in Hilprecht 1906: n°25; N 3891 in Sachs 1947: 234; 2N-T 500 in Robson 2000: 20; 3 NT-362 in Robson 2000: 22; Ni 10241 in Proust 2007 : §6.3.2; UET 6/2 295 in Friberg 2000: 101; MLC 651 in Sachs 1947: 233; YBC 1839 in Sachs 1947: 232; VAT 5457 in Sachs 1947: 234; TH99-T192, TH99-T196, TH99-T584, TH99-T304a are unedited tablets, soon to be published by A. Cavigneaux *et al.*; MS 2730, MS 2793, MS 2732, MS 2799 in Friberg 2007: § 1.4 (Note: among the tablets of the Schøyen collection published in this last work, are found other reciprocal pairs, but their reading presents some uncertainty).

[38] For example, format (1) is found on the obverse of the tablet Ni 10241, and format (2) on its reverse. (See the Appendix).

Number	Provenance	Contents	
		Reciprocal pairs	format
UM 29-13-021	Nippur	30 pairs: 2.5 / 28.48 etc. 6 pairs: 2.40 / 22.30 etc. 10 pairs: 1.40 / 36 etc. 8 pairs: 1.4 / 56.15 etc. 9 pairs: 4.3 / 14.48.53.20 etc.	(1)
TH99-T584, IV	Mari	1.4 / 56.15	(1)
TH99-T304a, IV	Mari	4.16 / 14.3.45 ($4.16 = 1.4\times2^2$)	(1)
VAT 5457, IV	Unknown	9.6.8 / 6.35.30.28.7.30 ($9.6.8 = 1.4\times2^9$)	(2)
MS 2730, IV	Unknown	4.51.16.16 / 12.21.34.37.44.3.45 ($4.51.16.16 = 1.4\times2^{14}$)	(2)
MS 2793, IV	Unknown	41.25.30.48.32 30 / 1.26.54.12.51.34.11.22.<1.52.30> ($41.25.30.48.32\ 30 = 1.4\times2^{23}$)	(1)
MS 2732, IV	Unknown	1.9.7.12 / 52.5 ($1.9.7.12 = 4.3\times2^{10}$)	(2)

Table 12.7: Reciprocal Exercises Not Appearing in Tablet A

Tables 12.6 and 12.7 show that a strong relation exists between Tablet A and the school texts. Nearly all the direct parallels (Table 12.6) or indirect parallels (Table 12.7) are Type IV school tablets. Each concerns a single reciprocal calculation. The tablets which are not of type IV contain lists of reciprocals, all like tablet A. These tablets are UM 29-13-021 and CBS 10201, from Nippur, as well as BM 80150, of unknown origin.

The majority of school exercises use the data found in Tablet A. Two exercises from Nippur are reproductions identical to Sections 9 and 10 of Tablet A, including the reverse sequence. When the factorization method is employed in the exercises, it uses the factors chosen in Tablet A, except in one case.[39] The tablets in Table 12.7 which do not use the geometric series with a common ratio of 2 and an initial number 2.5 still have links with Tablet A. Specifically, they use a geometric series with a common ratio of 2, but with an initial term of 1.4 (and in one case 4.3), as found for example in the tablet UM 29-13-021 from Nippur.

These observations could indicate that the tablets such as Tablet A and the other tablets which are not type IV school exercises (CBS 10201, UM 29-13-021, BM 80150) were the work of schoolmasters and that one of the purposes of their authors was the collection of exercises for the education of scribes. The link between Tablet A and teaching is incontestable, but does this signify that Tablet A is a ‘teacher’s textbook’ from which the exercises were drawn? Several arguments fit with this hypothesis, but it also raises serious objections. Beginning with what is now known about the school context and proceeding more specifically to Tablet A and its parallels I will present arguments for and against this text's being a ‘teacher’s textbook.’

The structure of school documents of an elementary level speaks in favour of the hypothesis. Lists of exercises can be considered a ‘teacher’s textbook’ if we consider only on this level. Exercises from the elementary level are extracts of texts written on tablets of a particular type, called ‘Type I’ by Assyriologists.[40] This relationship between a ‘teacher’s textbook’ and pedagogical extracts appears both for the mathematical texts as well as the lexical texts. However, as far as the advanced school texts are concerned, whether they are lexical or mathematical like the reciprocal exercises, the situation is different and far from simple. The exercises are not formulaic like those of an elementary level. If the documentation regarding the elementary level is composed of numerous duplicata, the documentation at an advanced

[39] In the tablet CBS 1020, the factorization of 16.40 uses the factor 40 in place of 6.40. It is not, however, a type IV school text, but a tablet containing a list of 8 reciprocals, the function of which is closer to the function of Tablet A.

[40] Some authors think that the type I tablets from Nippur are perhaps the product of students who have finished their elementary education, undergoing some type of examination (Veldhuis 1997: 29-31).

level is composed only of unique instances, and this is true for the lexical texts and for the mathematical texts. Duplicata occur neither among the advanced school exercises nor among the most erudite texts to which they are connected. The school documentation at an advanced level thus does not present as clear and regular a structure as that at an elementary level, and it cannot be relied on to identify the nature of the relationship which connects Tablet A with the school exercises.

Nevertheless, the important fact remains that Tablet A has a large number of pedagogical parallels. Moreover, the known school exercises about reciprocal calculations all bear upon a number connected with the data in Tablet A, whether directly (one of the terms of the series of doublings of 2.5), or indirectly (one of the terms of the series of doublings of another number such as 1.4 or 4.3). These instances have a unique relationship with the direct sequences on Tablet A. On the other hand, reverse sequences are rarely found in the school exercises. They appear only in two tablets from Nippur, which reproduce exactly the Sections 9 and 10 of Tablet A, and in a tablet from Mari (TH99-T196). Again, in the two cases from Nippur, the reverse sequences are not isolated, but associated with the direct sequences. Thus, it is not the data from the reverse sequences which provides the material for the school exercises, but rather the data from the direct sequences. In general, the reverse sequences provide a very small contribution to the prospective 'collection of exercises' for teaching, and yet they constitute half the text of tablet A.

The pedagogical interest in the series of doublings of 2.5 must also be considered because this series allows the repetition of the same algorithm many times, under conditions where it provides only results known in advance, with a gradually increasing level of difficulty. In fact, this argument relates to the educational value of the geometric series with a common ratio of 2 and an initial term of 2.5, not to Tablet A in its entirety. Tablet A is constructed around the idea of reciprocity, a notion clearly fundamental to it author and hardly present in the ordinary exercises about reciprocal calculations.

These considerations lead to the notion that it is possible that the relationship between Tablet A and the school exercises is exactly opposite of what is usually allowed. Tablet A does not seem to be the source of school exercises: rather it seems derived from the school materials with which the scribes of the Old Babylonian period were familiar. In this case, the material was developed, systematized and reorganized with different objectives than the construction of a set of exercises.[41]

The function of the reverse sequence seems to be the key to understanding the whole text. It has been suggested above that the reverse sequence might play a role in relation to the functional verification of the algorithm. The question which arises concerns, more precisely, the nature of the relationship between the direct sequence and the reverse sequence. In order to advance this inquiry, we turn to other cases in the cuneiform documentation which present direct and reverse sequences. As emphasized in the introduction, these cases appear in several tablets containing calculations of square roots. Thus let us examine these calculations.

SQUARE ROOTS

Sources presently known to contain calculations of square roots are not so numerous as those concerning reciprocals. Nonetheless, they present interesting analogies with what we have just considered. First of all, texts in both a numeric and verbal style are found for the same algorithm. Additionally, the fundamental elements of the reciprocal algorithm–factorization,

[41] This process may be compared to that described by J. Friberg for the various Mesopotamian and Egyptian texts under the name of 'recombination texts'. For him, this type of compilation is tightly connected with educational activity (Friberg 2005: 94).

spatial arrangement in columns (in the case of the numeric texts) and the presence of reciprocity–appear in these texts. This small collection of texts allows us to consider some of the problems raised above from other angles: the nature of the reciprocal algorithm, the connections between the direct and reverse sequences, the specificity of numeric texts with respect to verbal texts, and the nature of the links that the different types of texts have with education.

The following table gives the list of tablets containing the calculations of square roots (I recall in column 1 the letters indicated in Table 12.1). I have likewise included those which contain calculations of cube roots, though no numeric version occurs with cube root calculations. This absence poses an interesting question: is this the result of chance in preservation or a significant fact?

	Number	**Prov.**	**Calculation**	**Style**
C	UET 6/2 222, IV	Ur	Square root of 1.7.44.3.45 (result: 1.3.45)	Numeric
	3 N-T 611, IV	Nippur	Square root of 4.37.46.40 (result: 16.40)	Numeric
	HS 231, IV	Nippur	Square root of 1.46.40 (result: 1.20) (uncertain reading)	Numeric
	TH99-T3, IV	Mari	Square root of 2.6.33.45 (result: 11.15)	Numeric
	Si 428, IV	Sippar	Square root of 2.2.2.2.5.5.4 (result: 1.25.34.8)	Numeric
D	IM 54472	Unknown	Square root of 26.0.15 (result: 39.30)	Verbal
	YBC 6295	Unknown	Cube root of 3.22.30 (result: 1.30)	Verbal
	VAT 8547	Unknown	Cube roots of 27, 1.4, 2.5 and 3.36 (results: 3, 4, 5, 6 respectively)	Verbal

Table 12.8: Calculations of Square and Cube Roots[42]

Tablet D, of unknown origin, contains a text composed in Akkadian which concern the procedure of calculating the square root of 26.0.15. For a detailed analysis, see the various publications on the subject of this text.[43] Two interesting points should be highlighted here. The first is the presence of the factorization algorithm, in the form of instructions wherein the terms are quite similar to those in Tablet B regarding reciprocals. The second is the last phrase: ‘39.30 is the side of your square. 26.0.15 is the result (of the product of 39.30 by 39.30).’ The tablet thus ends with a verification of the result.

Tablet C is a small lenticular school tablet, the transcription and copy of which are as follows:[44]

[42] The tablets of Table 12.8 have been published in the following articles and works. C=UET 6/2 222 in Gadd and Kramer 1966: n°222 – see Table 12.1; YBC 6295 in Neugebauer and Sachs 1945: 42; VAT 8547 in Sachs 1952: 153; D=IM 54472 in Bruins 1954: 56 – see Table 12.1; TH99-T3 is an unedited tablet, soon to be published by A. Cavigneaux *et al.*; Si 428 in Neugebauer 1935-7 I: 80; HS 231 in Friberg 1983: 83; 3 N-T 611 in Robson 2002: 354; YBC 6295 in Neugebauer and Sachs 1945: text Aa, this tablet is believed to have come from Uruk, in the south of Mesopotamia according to Neugebauer 1935-7 I: 149 and to H2002: 333-337; VAT 8547 in Sachs 1952: 153.

[43] Chemla 1994: 21, Muroi 1999: 127, Friberg 2000: 110. Because no copy of the text has yet been published, it is not known if the presence of zero in the middle place is indicated on the tablet by a blank space, as sometimes happens in cuneiform texts, particularly those of the first millennium.

[44] Copy: Gadd and Kramer 1966; transcription: Friberg 2000: 108. See also Robson 1999: 252.

C

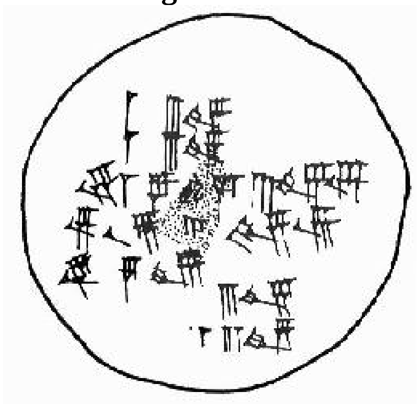
Transcription			Calculations	Copy
	1.3.45		1.3.45 × 1.3.45 = 1.7.44.3.45	**Figure 12.2** 
	1.3.45			
15	1.7.44.3.45	16	inv(3.45) = 16 ; sq.rt.(3.45) = 15	
15	18.3.45	16	inv(3.45) = 16 ; sq.rt.(3.45) = 15	
17	4.49		sq.rt.(4.49) = 17	
	3.45		15 × 15 = 3.45	
	1.3.45		3.45 × 17 = 1.3.45	

Table 12.9: Tablet C

The process of calculation by factorization occurs in the case of Tablet C, as J. Friberg has remarked. The number 1.7.44.3.45 ends with 3.45, which is selected as an elementary regular factor. The number 16, its reciprocal, is set out on the right; on the same line, the number 15, its square root, is set out on the left; the product of 1.7.44.3.45 by 16 (which gives a second factor) is placed on the centre of the following line. The process is repeated until a number for which the square root is given by the standard tables is found.[45] The desired square root is the product of the numbers recorded on the left.

It should be noted that this small text, like those found in the sections of Tablet A, begins and ends with the same number, and as before, the calculation forms a loop. It starts with an arithmetical operation (the square of 1.3.45), then it proceeds by a sequence which carries out the reverse operation (the square root of the resulting number, 1.7.44.3.45). Here, the direct sequence and the reverse sequence rely on algorithms of a different nature, even though in the cases involving reciprocals, they rely on the same algorithm. Could it be said that the calculation of the square of 1.3.45 is a simple verification of the result of the calculation of the square root? In this case, it would be logical that the verification come at the end of the calculation (as is the case in the verbal Tablet D) and not at the beginning. The text thus illustrates something else, which seems to relate to the fact that the square and the square root are reciprocal operations. This ‘something else’ is perhaps akin to what the author of Tablet A illustrated with the reverse sequences.

The algorithm for calculating square roots is based on the same mechanism of factorization as that for determining the reciprocal. In the numeric versions, the rules concerning the layout are analogous: the factors are placed in the central column; the reciprocals of these are placed to the right; a supplementary column appears on the left, in which are placed the square roots of the factors. This supplementary column shows us that the algorithm in fact has two components: a factorization (right column) and square root (left column). In the case of the reciprocal’s algorithm, the right-hand column provides the factors which serve all at once as the factorization and the determination of the reciprocals. Thus the two components merge. However, the method of application of the factorizations presents a particular mathematical problem for the square roots. In effect, the algorithm for finding a reciprocal is, by definition, applied to the regular numbers. The factorizations are always possible, and lead mechanically

[45] As in the case of the reciprocals, the calculations of the squares and square roots rely on a small stock of basic results memorized by the scribes during their elementary education. The tables of squares and square roots are largely found in the school archives. See, notably, Neugebauer 1935-7 I: ch. I.

to the result. Alternately, perfect squares can quite easily be the product of irregular numbers, and in this case, factorization by the standard method is impossible. The important point to note is that, even though the algorithms for the determination of the reciprocal and the extraction of a square root diverge from one another in their components and even though they present different mathematical problems as their topic, they are presented in the texts in a parallel fashion.

The specificity of the numeric texts with regard to the verbal texts thus appears more clearly. For the square roots, the layout of the numeric texts observes the same rules regarding arrangement in columns as for the determination of reciprocals. This spatial arrangement facilitates control of the calculation. In fact, it is enough, when finding the desired number, to multiply all those which are set out on the right in the case of reciprocals, and those on the left in the case of roots. It is notable that, in the case of reciprocals as well as square and cube roots, the verbal versions contain only numbers of a small size, which do not demand recourse to iteration. The numeric versions contain numbers of large size, and the arrangement in columns shows that it is possible to develop the iterations without limit, which confers power on the process.

The verbal and numeric versions of the calculations of square roots refer nonetheless to the same algorithms. In fact, the verbal texts contain instructions which detail how to ‘place’ certain numbers ‘beneath’ others, in a way which corresponds with the spatial arrangement of the numeric texts.

What is the place of square roots in the education of the scribes? The format of the tablets of the calculation of square roots, which are all of Type IV for the numeric versions, shows that they were school exercises. However, in this case the exercises are much less standardized than the calculation of reciprocals. For square roots, the numeric repertory offers no regularity, whereas for the reciprocals, the repertory is homogeneous (as seen above, it is based principally on the doublings of 2.5). Moreover, the group of tablets containing the calculations of square roots is small, whereas the group of exercises of the calculation of reciprocals is numerically important. The great frequency of calculations of reciprocals is undoubtedly explained by the importance of this technique in scientific calculation, but another reason may be postulated. In the reciprocal, the two components (factorization and the determination of reciprocals) are superimposed. The algorithm for the determination of a reciprocal by factorization puts the mechanism of factorization first. The determination of a reciprocal by factorization is thus a fundamental procedure,[46] essential to other algorithms, even though it is applied in a less general way for the roots than for the reciprocals. Consequently, the reciprocal exercises probably occupy a more elementary educational level than those which contain square roots. The calculations of square roots may be situated between the work of beginning scribes and works of scholars, in a gray area that has left us few traces.

What, then, of the cube roots? They appear in two verbal texts, wherein they are treated in a manner identical to the square roots, except for the verifications, which do not appear in either case.[47] No numeric version is known for these calculations. It cannot be excluded that the absence of a numeric version of the calculation of a cube root is due to the chances of

46 The Akkadian term *makṣarum* probably has some link with the process of factorization. It appears in two texts, in slightly different senses: it appears in the *incipit* of tablet YBC 6295 cited in table 12.6 ([*ma*]-*ak-ṣa-ru-um ša* ba-si = the *makṣarum* of the cube root); it designates an enlargement in tablet YBC 8633.

47 Note also the following curious detail: in the VAT 8547, all the entries appear in the standard tables of cube roots, and the application of the reciprocal algorithm to these numbers leads to a complication of the situation. Thus, 27 is decomposed according to a somewhat artificial manner as the product of 7.30 and 3.36. It is clear that in this case, as in that in Tablet A, the purpose is not to obtain a new result.

preservation but other explanations are possible. Indeed, the tables of squares, square roots and cube roots are known to us from the preserved numeric tablets, but tables of cubes are unknown. The absence of a table of cubes is undoubtedly linked to the fact already mentioned that multiplication is an operation with two arguments. Consequently, the cube root has no reverse operation in the Mesopotamian mathematical tradition. This fact would explain why it has not been found in a numeric format, which is founded on the notion of reciprocity.

This analysis of the calculation of square roots also emphasizes by contrast the fact that the reciprocal algorithm is a combination of two different components (factorization and the determination of a reciprocal). In addition, it may be seen that the numeric texts have an approach relatively unified with that of the reciprocal algorithm. The function of the reverse algorithm seems the same in all cases. It does not enact a verification of the result, or even a verification of the algorithm itself in the case of the square roots, since the direct and reverse sequences do not rely on the same algorithm. Their presence seems to indicate something else with respect to the nature of the operations themselves. It stresses the fact that the reverse operation of a square is the square root, and the reverse operation of the reciprocal is the reciprocal itself.

CONCLUSION

I can now reconsider several questions left aside from the preceding discussion. The function of the tablet is at the heart of these questions, and I will treat these questions before returning to the ways of reasoning we can detect in the text.

It has been seen that the content of Tablet A is connected to the context of teaching but that it cannot be interpreted as a simple collection of data intended to provide exercises for the education of young scribes. I have suggested that its relationship with the school exercises could be the reverse of what is generally supposed. It might not be a ‘teacher’s textbook’ from which the school exercises were taken but rather a text constructed and developed from existing school material. Indeed the relationships between school exercises and scholarly texts were probably not so uni-directional and the two relations could well be combined. However, the point which interests us here is that the Tablet A appears in the form of an original inquiry and its purpose seems to have been communication between erudite scribes. Seen from this perspective, the same piece of text takes on another dimension. The way in which the text is organized and arranged, and the repertory of numeric data on which it is built are essential components of the text. In a certain way, these components constitute the means of expression by which Tablet A refers to the reciprocal algorithm.

But what is the relationship between Tablet A and the algorithm for reciprocal? Is it a practical text in the sense that the text executes concretely the operations necessary for the determination of a reciprocal? It is not certain that the writing of a text was essential to the execution of the algorithm, since the known texts obviously record only part of the series of actions which allow the result to be obtained. On the one hand, the multiplications are probably executed elsewhere. On the other hand, by the standards of school practices, the written traces are incomplete. They often state only the first step of the process of factorization, as is notably the case in the tablets of the *Schøyen Collection* published by J. Friberg listed in table 12.7. The tablet does not refer to all the steps necessary to execute the algorithm. Text A is not a simple set of instructions for execution of the reciprocal algorithm.

What does tablet A say about this algorithm and how?

First of all, the author of Tablet A expresses himself by means of numbers arranged in a precise way, not by means of a linear continuation of the instructions, as is done in the verbal texts. The numeric texts refer to the same algorithms as the verbal texts, but they do it in a different way. The spatial arrangement of the writing has its own properties and emphasizes certain functions of the algorithm. The arrangement into columns renders the process of determining a reciprocal transparent. Indeed, to find the desired number, it is enough to multiply the numbers on the right in the case of the reciprocals and the numbers on the left in the case of the roots. The arrangement into columns certainly recalls the practices of calculation external to the text, but the fact that this arrangement was set in writing clearly emphasizes the principles of the function of the algorithm – that is, the fact that it is possible to factorize the regular numbers into the product of regular numbers and the fact that the reciprocal of a product is the product of the reciprocals. Moreover, the spatial arrangement of the text underscores the power of the procedure of developing the iterations without limitation. On this topic, let us recall the striking fact that the recourse to iteration does not appear in the verbal texts, which limit themselves to numbers of a small size, whereas the iteration expands in a rather spectacular way in Tablet A, and in a more modest way in the numeric versions of the calculations of the square roots. For the ancient reader, the spatial arrangement of the numbers in Tablet A serves the functions that Sachs' formula does for the modern reader: it shows why the algorithm works. The layout says more than the formula in showing not only why, but also **how** it operates and what its power is.

Tablet A is constructed on the continuation of the doublings of 2.5. The educational value of this series in the instruction of the factorization algorithm has been underscored above, but perhaps the essence lies elsewhere. The fact that the scribes limited themselves to the geometric series with an initial number 2.5 and a common factor of 2 guarantees the regularity of the entries. This series assures the calculator that the result remains in the domain of regular sexagesimal numbers, a condition necessary for the existence of a sexagesimal reciprocal (with finite expression) and for the operation of the algorithm. It undoubtedly did not escape the scribes that it was possible to choose other series. (In tablet UM 29-13-021 are found series based on other initial terms, such as 2.40, 1.40, 4.3). However, the series of doublings of 2.5 is a typical example which allows the scribes to refer to the algorithm by specific numeric data. In other words, this series plays the role of a paradigm. It is possible that the choice of 2.5 comes from the previously noted fact that this number is a logical continuation of the standard reciprocal tables in which the last entries are 1.4 and 1.21.

Fundamentally, Tablet A is built on reciprocity. What expresses the regular and systematic presence of the reverse sequences? It has been shown that the purpose was not the verification of the results because such a matter could have taken a much simpler form. It could have had a role in the verification of the algorithm itself and thus ensured the validity of the mechanism. However, as suggested above, the significance of the reverse sequences could have been above all to express a mathematical rule: 'the reverse of the reverse is itself'. Whatever the case may be, it is clear that in the reverse sequences, the author abandons the stereotypical patterns found in the direct sequences of the text (and found also in the school exercises) and plays with the freedom remaining to him in the choice of factors for the decomposition into elementary regular factors. The reverse sequences thus highlight another important mathematical aspect: the multiplicity of decompositions.

The purpose of the text on Tablet A is thus clearly the algorithm itself, its operation and its justification. The text refers to the algorithm not in a verbal manner, but by an interpretable spatial arrangement, the exploitation of a paradigm well known to the scribes, and the recourse to the reverse sequences in a systematic way. Tablet A therefore bears witness to the reflection of the ancient Mesopotamian scribes on some of the fundamental principles of numeric calculation: the possibility of decomposing the regular numbers into two or more (through iteration) elementary regular factors, the freedom which the multiple valid decompositions offer to the calculator (given that the direct and reverse sequences show two different strategies for the selection of factors), the stability of the multiplication for reciprocal (the reciprocal of a product is the product of reciprocals of the factors), the involutive character of the determination of a reciprocal (given the fact that the reciprocal is its own reverse operation).

Translation Micah Ross

APPENDICES

Appendix 1 : Tablet A (CBS 1215)

Sachs 1947: 237; Robson 2000: 23. The asterisks recall the remark which follows the transcription. I added the elements of the appearance to facilitate the reading: the final part of the number which plays a role as a factor is set in bold; the final result of the calculation is underlined; the format reproduces the layout of the tablet.

Table 12.10

Obverse

Column I	
#1	
2.**5**	12
25	2.24
28.**48**	1.15
36	1.40
2.5	
#2	
4.**10**	6
25	2.24
14.**24**	2.30
36	1 .40
4.10	
#3	
8.**20**	3
25	2.24
7.**12**	5
36	1.40
8.20	
#4	
1**6.40**	9
2.30	24
3.[**36**]	[1.40]
6	10
15^{sic}.40	
#5	
3**3.20**	18
10	6
1.**48**	1.15
2.**15**	4
8^{sic}	6.40
26.40	
33.20	
#6	
1.**6.40**	9
10	6
54	1.6.40
#7	
[2].1**3.20**	18
[40]	1.30
[**27**]	2.13.20
#8	
4.2**6.40**	9
40	1.30
13.**30**	2
27	2.13.20
4.26.40	

Column II	
#9	
8.5**3.20**	18
2.40	22.30
6.**45**	1.20
9	6.40
8.53.20	
#10	
17.4**6.40**	9
2.40	22.30
3.22.**30**	2
6.**45**	1.20
9	6.40
8.53.20	
17.46.40	
#11	
36^{sic}.2^{sic}**3.20**	18
10.**40**	1.[30]
[**16**]	3.4[5]
5.37.30	
[1.41.**1]5**	4
[6.**45**]	1.20
[**9**]	6.40
[8.53].20	
[35.33].20	
#12	
[1].11.**6.[40]**	9
10.**40**	1.[30]
16	3.4[5]
5.37.30	
50.37.**30**	2
1.41.**15**	4
6.**4[5]**	1.20
9	6.40
[8.5]3.[20]	
35.33.20	
1.11.6.40	
#13	
2.22.1**3.20**	[18]
4**2.40**	22.30
16	3.45
1.24.22.30	
25.18.**45***	[16]

Column III	
6.**45**	1.20
9	[6.40]
8.53.20	
[2.2]2.13.[20]	
#14	
4.44.2**6.40**	[9]
4**2.40**	2[2.30]
16	3.[45]
1.24.22.30	
[12.3]9.22.**30**	[2]
[25.18].**45***	[16]
[6.**45]**	[1.20]
[9]	[6.40]
[8].53.20	
[2.22.13. 20]	
[4.44.26.40]	
#15	
[9.28].5**3.[20]**	[18]
2.50.**40**	[1.30]
[4.**16]**	[3.45]
[16]	[3.45]
14.3.[45]	
[2]1.5.3[7.30]	
[6.19.4]1.**15**	[4]
[25.18.**45**]*	[16]
[6.**45]**	[1.20]
[9]	[6.40]
[8.53.20]	
2.[22.13.20]	
9.[28.53.20]	
#16	
18.57.[4**6.40**]	[9]
[2.50.**40]**	[1.30]
4.[**16**]	[3.45]
16	[3.45]
[14].3. [45]	
[21.5.37.30]	
[3.9.50.37.**30**]	[2]
[6.19.41.**15**]	[4]

(continued on the reverse)

Reverse (on the reverse of the tablet, the columns run from right to left, as is customary.)

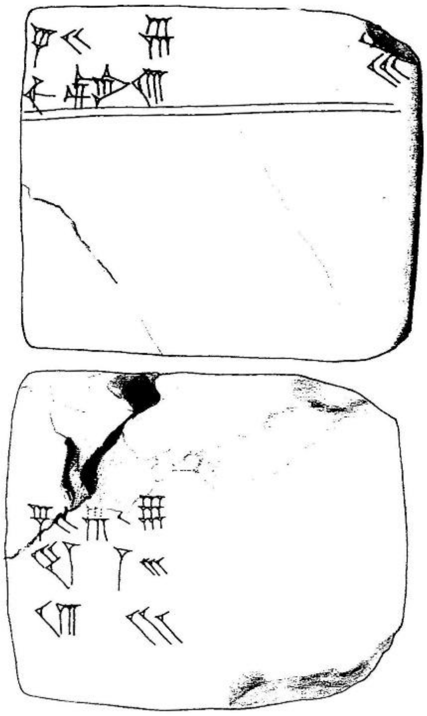
Column III	Column II	Column I
#21 10.6.48.5**3.20** 18 3.2.2.**40** 22.[30] 1.8.**16** 3.4[5] 4.**16** 3.[45] **16** 3.[45] 1[4.3.4]5 52.44.[3.4]5 19.46.31.24.22.[30] <u>5.55.57.25.18.**4[5]**</u> 16 1.34.55.18.**45*** 16 25.18.**45*** [16] 6.**45** [1.20] **9** [6.40] 8.53.20 2.22.13. 20 37.55.33.20 10.6.48.53.20	#19 [2.31.42.1**3.20** 18] [45.30.**40** 1.30] [1.8.**16** 3.45] [4.**16** 3.45] **16** [3.45] 14.[3.45] 5[2.44.3.45] 1.18[sic].6.[5.37.30] <u>23.43.49.[41.**15**]</u> [4] 1.[3]4.55.18.**45*** [16] [25].18.**45*** 1[6] [6].**45** 1.[20] **[9]** 6.40 8.53.20 2.22.13.20 37.55.3[3.20] 2.31.42.13.[20] #20 5.3.24.2**6.40** [9] 45.30.**40** 1.30 1.8.**16** 3.45 4.**16** 3.45 **16** 3.45 14.3.45 5[2.44].3.45 1.19.6.5.37.30 <u>11.51.54.50.37.**30**</u> 2 23.43.49.41.**15** 4 1.34.55.18.**45*** 16 25.18.**45*** 16 6.**45** 1.20 **9** 6.[40] 8.53.20 2.22.13. 20 37.55.33.20 2.31.42.13.20 5.3.24.26.40	#16 (continued) [25.18.**45*** 16] [6.**45** 1.20] **[9** 6.40] [8.53.20] [2.22.13. 20] [9.28.53.20] [18.57.46.40] #17 [37.55.3**3.20** 18] [11.2**2.40** 22.30] [4.**16** 3.45] [**16** 3.45] [14.3.45] [5.16.24.22.30] [<u>1.34.55.18.**45***</u> 16] [25.18.**45*** 16] [6.**45** 1.20] **9** [6.40] [8.53.20] 2.22.13.[20] 37.55.33.[20] #18 1.15.51.**6.40** 9 11.2**2.40** 22.30 4.**16** 3.45 **16** [3.45] 14.[3.45] 5.16.[24.22.30] <u>47.27.[39.22.**30**</u> 2] [1.34.55.18.**45*** 16] [25.18.**45*** 16] [6.**45** 1.20] **[9** 6.40] 8.[53.20] 2.2[2.13. 20] 37.55.[33.20] 1.15.51.[6.40]

Notes
#4: Read 16.40 in place of 15.40.
#5: Read 9 in place of 8.
#11: Read 35.33.20 in place of 36.23.20.
#19: Read 19 in place of 18.

***Remark**
#13 to #21: The factor chosen is 3.45 (from the reciprocal 16). I could not set it in bold type because it does not obviously constitute the final part of the number, as in the other cases. However, if 8 is decomposed into the sum 5+3, the factor 3.45 is scarcely hidden. (For more precise details, see the part of the article devoted to the analysis of the entirety of this text.)

Appendix 2 : Ni 10241

Old Babylonian School Tablet from Nippur, Conserved in Istanbul, Copy Proust 2007.

<table>
<tr>
<td>
Figure 12.3</td>
<td>**Obverse**
4.26.[40]
its reciprocal 13.30

Reverse
4.26.40 9
41[sic] 1.30
13.30</td>
</tr>
</table>

Table 12.11

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